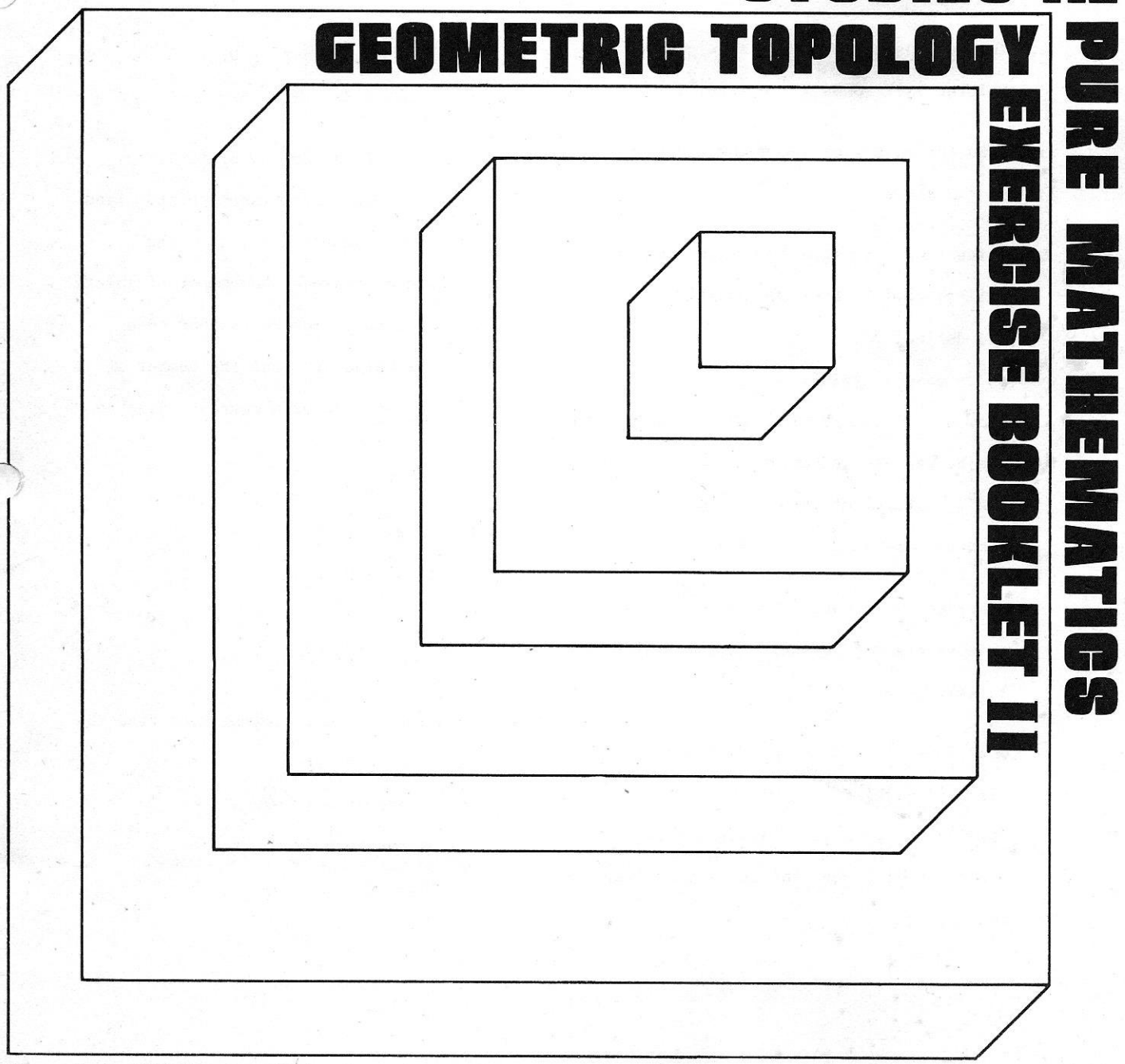
M335 GT EBI

THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M335 Studies in Pure Mathematics

STUDIES IN PURE MATHEMATICS

GEOMETRIC TOPOLOGY

EXERCISE BOOKLET II



Copyright © 1983 The Open University

SUP 10412 9

1.3

Section 5.1

1. Plot each of the following numbers in the plane of complex numbers:

   $z = 2$; $z = 0$; $z = i$; $z = 1 + i$; $z = 1 - i$;

   $z = 2 - i$; $z = -2$; $z = -2 - i$, $z = -2 + i$.

2. Show that the inverse of $x + yi$ is $\frac{x - yi}{x^2 + y^2}$.

3. For each of the numbers $z = x + yi$ in (1) above, plot their complex conjugate $\bar{z} = x - yi$. Observe that z is real if and only if $\bar{z}$ is real ($y = 0$).

4. The modulus of $z = x + yi$ is $(x^2 + y^2)^{1/2}$, written $|z|$. Show $z^{-1} = \bar{z}/|z|^2$.

5. Find the moduli of the numbers $z = x + yi$ in (1) above.

6. Write $z = x + yi = r(\cos\theta + i \sin\theta)$ measuring θ anti-clockwise from the positive x-axis, and define

   $e^{i\theta} = \cos\theta + i \sin\theta$.

   Show $e^{in\theta}$ $(= \cos n\theta + i \sin n\theta) = (\cos\theta + i \sin\theta)^n$ can be used to deduce formulae for $\cos^n\theta$ or $\sin^n\theta$ in terms of $\cos\theta$ and $\sin\theta$.

   (Concentrate on $n = 2$ or 3.)

7. Describe the behaviour of z^{-1}, $\bar{z}$, and $\bar{z}^{-1}$ as z traces out the unit circle ($|z| = 1$) anticlockwise.

8. Show $|a + b| \leq |a| + |b|$, and deduce

   (i) $|a| - |b| \leq |a - b|$,

   (ii) $|a| - |b| \leq |a - b| \leq |a| + |b|$.

   Interpret (ii) when $|b|$ is much smaller than $|a|$.

9. For each of the following loci write down the number of points in the locus that correspond to a general point on the z-sphere.

   (i) $w^2 = z$

   (ii) $w^3 = z$

   (iii) $w^4 = z$

   (iv) $w^2 = z^2 + z$

   (v) $w^2 = z^3 + z$

   (vi) $w^3 = z^2 + 2z$

   (vii) $w^4 = z^3$

   (viii) $w^5 = z^7$

   (ix) $w^3 + 2wz + z^2 = 0$

   (x) $w^3 + 2wz^5 + z^6 = 0$

10. For each of the following loci, $(z,w) = (0,0)$ is a branch point. Find the branch point order, i.e. the difference between the number of points on the locus which correspond to a general value of z and the number of distinct values of w corresponding to $z = 0$.

    (i) $w^2 = z$

    (ii) $w^3 = z$

    (iii) $w^7 = z$

    (iv) $w^3 = z^2$

    (v) $w^3 = z(z + 2)$

11. For each of the following loci find the number of points

    (a) of the form (∞,w), $w \neq \infty$

    (b) of the form (z,∞), $z \neq \infty$

    (c) (∞,∞)

on the surface, counted according to multiplicity.

(i) $w^2 + 2wz + 2 = 0$

(ii) $3w^2 + 2z^2 + 1 = 0$

(iii) $w^2 + z^3 = 0$

(iv) $w^4 + 2w^2z + z^3 = 0$

(v) $w^3 + 2w^2z + 4wz^2 + z^3 = 0$

(vi) $w^4 + 3wz + wz^2 + 2 = 0$

(vii) $w^3z + 3wz + wz^2 + 1 = 0$

12. For each of the following loci find the number of finite points corresponding to $z = 0$, $z = 1$, and a general value of z.

(i) $z^2w^3 - w^2 - 2zw + z^5 + 1 = 0$

(ii) $(z - 1)w^2 - z^3w + z^2 - 2 = 0$

(iii) $z(z - 1)w^2 - zw - z + 1 = 0$

(iv) $(z^2 - 1)w^3 - z^2w^2 + 3z = 0$

(v) $(z - 2)w^4 - z^2w^2 + 2 = 0$

(vi) $(z + 1)w^4 - z = 0$

Sections 5.2 and 5.3

1. Show that each of the following quadratics is reducible, and reduce them.

(i) $w^2 + 3wz + 2z^2 + z - 1 = 0$

(ii) $2w^2 + 3wz - 2z^2 - 2w + z = 0$

(iii) $w^2 - wz - w - z - 2 = 0$

(iv) $w^2 - 3wz + 2z^2 - w + 2z = 0$

(v) $2wz + w - 4z - 2 = 0$

2. Analyse each of the following conics, locate their branch points and pinch points (if any). (Either work each example from first principles or quote the main theorem of the unit, as you see fit. We shall expect you to be able to work examples from first principles for assessment and examination purposes.)

(a) $wz + 2z + 3 = 0$

(b) $z^2 + wz + 4w + 1 = 0$

(c) $z^2 - w^2 + 2w + z = 0$

(d) $z^2 + 2zw + w^2 - z + 3 = 0$

(e) $6zw + w^2 - z + 2w + 3 = 0$

(f) $4zw + w^2 - 4z + 4 = 0$

(g) $2z^2 + zw - w^2 + 3 = 0$

(h) $z^2 - 2wz + z - 2 = 0$

(i) $z^2 + 2wz + w^2 + w + 2 = 0$

(j) $zw + w^2 + z = 0$

(k) $z^2 - 3zw + w^2 - 4 = 0$

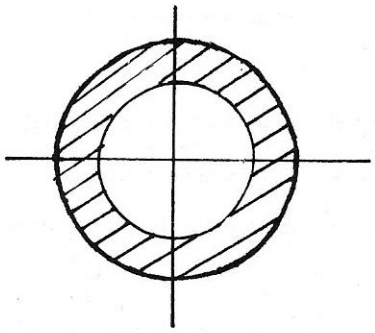
Remark: The essential topological point is that the locus is either a sphere and the map $(z,w) \mapsto z$ is a homeomorphism, or (and this is the more general case) the locus is a sphere (possibly pinched) and the map $(z,w) \mapsto z$ is 2-to-1 except at two points, called the branch points, and at the pinch point if it exists. We say the locus is a branched 2-fold cover of the z-sphere. This simple relationship between the locus and the z-sphere will be generalized in Unit 6, when it will emerge that more complicated loci than quadratics give rise to more interesting objects than spheres. Indeed, as we shall see, the covering space idea applies to any pair of surfaces with a (typically) n-to-1 map between them, so it is more general than loci.

Section 6.1

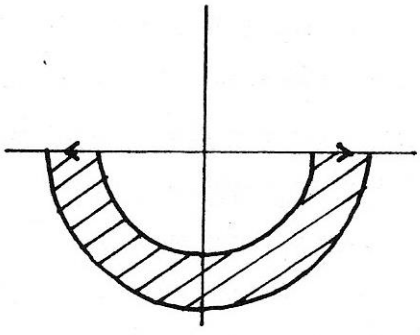
We give two examples of symmetries of surfaces with boundary which yield covering surfaces.

Example 1

Consider the annulus $A = \{z \in \mathbb{C}: 1 \leq |z| \leq 2\}$ and the action of a rotation by π about the origin: $z \mapsto -z$ $\quad (z \in A)$.



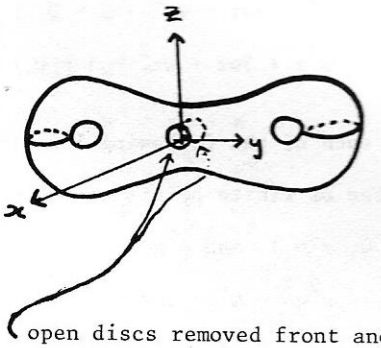
Identifying each point with its image very nearly gives the southern half of the annulus, $A' = \{z \in A: \Im m(z) \leq 0\}$, but the points x and -x must be identified. So the surface we seek is A' with the two horizontal edges identified:



The result is also an annulus. (Thus we have recaptured the example of an annulus double covering an annulus in Unit 5.)

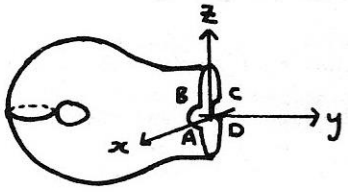
Example 2

Let S be a double torus with two discs removed which are symmetrically situated as shown:



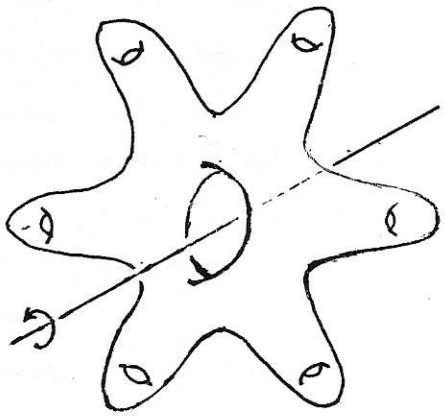
open discs removed front and back

Again we consider a rotation through π (around the x-axis, this time). The result of identifying each point with its image is almost presented by this surface, S':



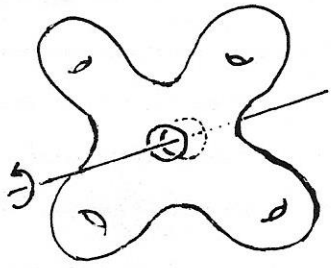
But points (x, 0, z) and (x, 0, -z) of S' lying in the plane y = 0 must be identified. In particular, A is to be identified with B and C with D. The result is a torus with 2 discs removed.

1. Let $\tilde{S}$ be a 7-holed torus and consider rotations through $2\pi/3$ of S (thought of as symmetrically situated about an axis.)



Show that this gives a threefold covering by $\tilde{S}$ of a three-holed torus.

2. Let $\tilde{S}$ be a 4-holed torus with 2 holes symmetrically situated about an axis. Consider the rotations through 0, $\pi/2$, π and $\frac{3\pi}{2}$ around the axis. Show that this gives a four-fold cover of a torus with 2 holes by $\tilde{S}$.

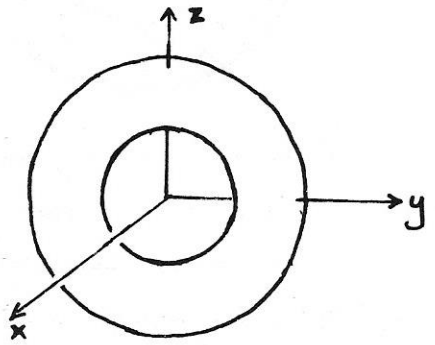


3. Modify Q1 to deal with the case of a 6-holed torus with two holes as a three-fold cover of a surface S, and describe S.

4. Show that an unbranched double cover of a torus with n-holes can always be produced from a torus with an odd number of holes, and display the covering geometrically.

Cautionary Note

It can happen that symmetries of a surface without boundary, $\tilde{S}$, lead to a surface with boundary, S. In this case $\tilde{S} \to S$ is not an example of a covering surface. This is illustrated by the following symmetries of a torus.



The torus is symmetrically situated about the coordinate origin and can be rotated about the x-axis. The symmetries in question are:

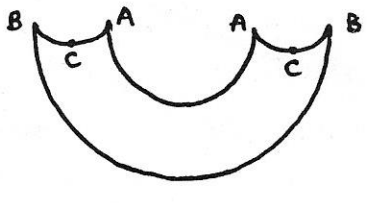
$(x, y, z) \xmapsto{e} (x, y, z)$, the identity

$(x, y, z) \xmapsto{\rho} (x, -y, -z)$, rotation through π about the x-axis

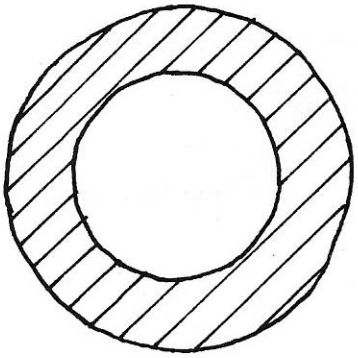
$(x, y, z) \xmapsto{\nu} (-x, -y, -z)$, the antipodal map

$(x, y, z) \xmapsto{\sigma} (-x, y, z)$, $\sigma = \rho\nu = \nu\rho$.

The resulting surface is almost the front lower half, since every point is equivalent to one with $x \geq 0$ (use σ if need be) and with $z \leq 0$ (use ρ if need be).



The arcs A C B must also be identified, however, by ρ, so the result is an annulus.



This space has two boundary components, whereas the torus has none, so there cannot be a local homemorphism between them. Consequently it is an example of something which is <u>not</u> a covering space.

Section 6.2

1. Show that any n-fold covering of a sphere ($n > 1$) must have at least two branch points.

2. Show that any 2-fold covering of a sphere by a sphere must have two branch points.

3. Show that the total branch point order, δ, of any covering of the sphere must be even.

4. Generalise Q3 above to any covering of a Riemann surface.

5. Show that none of the following covering spaces can occur.

   (a) $T^2 \xrightarrow{3} S^2$ unbranched

   (b) $S^2 \xrightarrow{3} T^2$ unbranched

   (c) $2K^2 \xrightarrow{2} T^2$

   (d) $4T^2 \xrightarrow{2} 2T^2$ unbranched

   (e) $K^2 \xrightarrow{2} T^2$

   (f) $4T^2 \xrightarrow{2} 3K^2$

   (g) $5T^2 \xrightarrow{3} 3K^2$

   (h) $3D^2 \xrightarrow{3} 2D^2$ unbranched

   (i) $4D^2 \xrightarrow{3} 3D^2$

   (j) $3M^2 \xrightarrow{4} 2D^2$

   (k) $4T^2 \xrightarrow{3} 2T^2$ branched

   (l) $9T^2 \xrightarrow{2} 5T^2$ branched

   (m) $7D^2 \xrightarrow{3} K^2 \# 2D^2$

6. Let $\tilde{S} \xrightarrow{n} S$ be an n-fold (possibly branched) covering of an orientable surface. Suppose $\tilde{S} = \tilde{k}T^2 \# \tilde{b}D^2$, and $S = kT^2 \# bD^2$, and $k \geq 1$. Show $\delta \leq 2\tilde{k} - 2$.

Section 6.3

It should be pointed out very firmly that different permutations of the edge symbols in n copies of an edge equation can well lead to different covering surfaces. For example, if you work through these two 3-fold coverings of K^2 you will find you get a branched covering by $4\mathbb{RP}^2$ and an unbranched covering by K^2 itself. Which is which?

K^2 is given by $aba^{-1}b = 1$.

(i) S_1 is given by $a_1 b_1 a_2^{-1} b_2 = 1$;
$a_2 b_2 a_1^{-1} b_3 = 1$;
$a_3 b_3 a_3^{-1} b_1 = 1$.

(ii) S_2 is given by $a_1 b_1 a_2^{-1} b_2 = 1$;
$a_2 b_2 a_3^{-1} b_1 = 1$;
$a_3 b_3 a_1^{-1} b_3 = 1$.

1. What 4-fold coverings of K^2 are obtained from these edge equations?

(i) $$\left.\begin{aligned} a_1\, b_1\, a_2^{-1}\, b_4 &= 1 \\ a_2\, b_2\, a_3^{-1}\, b_3 &= 1 \\ a_3\, b_3\, a_4^{-1}\, b_2 &= 1 \\ a_4\, b_4\, a_1^{-1}\, b_1 &= 1 \end{aligned}\right\} S_3$$

(ii) $$\left.\begin{aligned} a_1\, b_1\, a_2^{-1}\, b_2 &= 1 \\ a_2\, b_2\, a_1^{-1}\, b_3 &= 1 \\ a_3\, b_3\, a_4^{-1}\, b_4 &= 1 \\ a_4\, b_4\, a_3^{-1}\, b_1 &= 1 \end{aligned}\right\} S_4$$

(iii) $$\left.\begin{aligned} a_1\, b_1\, a_2^{-1}\, b_1 &= 1 \\ a_2\, b_2\, a_3^{-1}\, b_2 &= 1 \\ a_3\, b_3\, a_4^{-1}\, b_3 &= 1 \\ a_4\, b_4\, a_1^{-1}\, b_4 &= 1 \end{aligned}\right\} S_5$$

2. What are the following covering surfaces of the Möbius band with edge equation $aabcb^{-1} = 1$?

(i) $$\left.\begin{aligned} a_1\, a_1\, b_1\, c_1\, b_2^{-1} &= 1 \\ a_2\, a_1\, b_2\, c_2\, b_1^{-1} &= 1 \end{aligned}\right\} S_6$$

(ii) $$\left.\begin{aligned} a_1\, a_2\, b_1\, c_1\, b_1^{-1} &= 1 \\ a_2\, a_1\, b_2\, c_2\, b_2^{-1} &= 1 \end{aligned}\right\} S_7$$

3. What are the following covering surfaces of S, a Möbius band with a hole, with edge equation $cafabded^{-1} = 1$?

(i) $$\left.\begin{aligned} c_1\, a_1\, f_1\, a_2\, b_1\, d_1\, e_1\, d_2^{-1} &= 1 \\ c_2\, a_2\, f_2\, a_3\, b_2\, d_2\, e_2\, d_3^{-1} &= 1 \\ c_3\, a_3\, f_3\, a_1\, b_3\, d_3\, e_3\, d_1^{-1} &= 1 \end{aligned}\right\} S_8$$

(ii) $$\left.\begin{aligned} c_1\, a_1\, f_1\, a_2\, b_1\, d_1\, e_1\, d_1^{-1} &= 1 \\ c_2\, a_2\, f_2\, a_3\, b_2\, d_2\, e_2\, d_2^{-1} &= 1 \\ c_3\, a_3\, f_3\, a_1\, b_3\, d_3\, e_3\, d_3^{-1} &= 1 \end{aligned}\right\} S_9$$

4. Find the Riemann surfaces corresponding to each of the following loci.

(i) $w^5 = z(z - \alpha)$

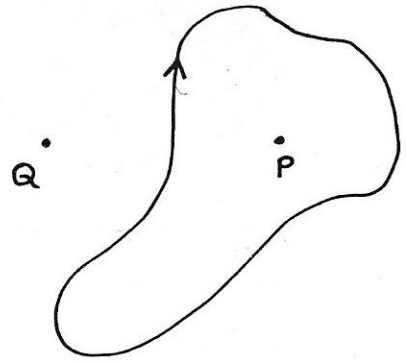
(ii) $w^4 = z(z - \alpha)(z - \beta)$

(iii) $w^3 = z(z - 1)(z - 2)(z - 3)$

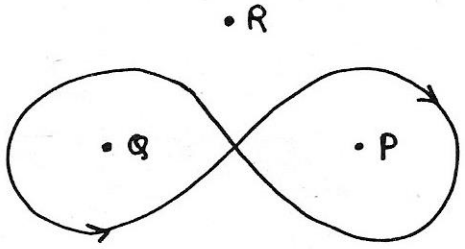
Section 7.1

1. Find the winding numbers of each of the following curves about the indicated points:

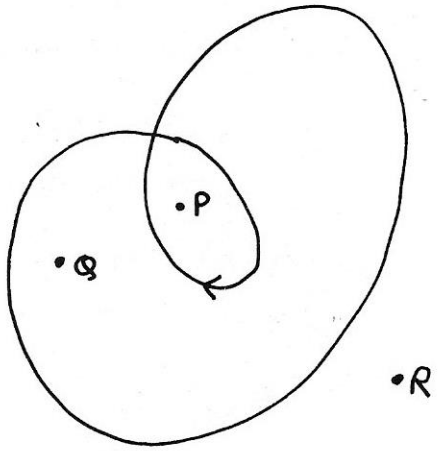
(i)



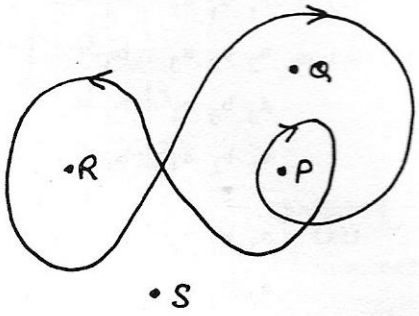
(ii)



(iii)

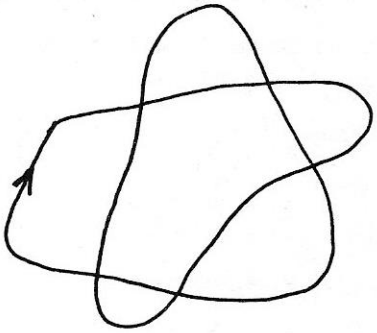


(iv)

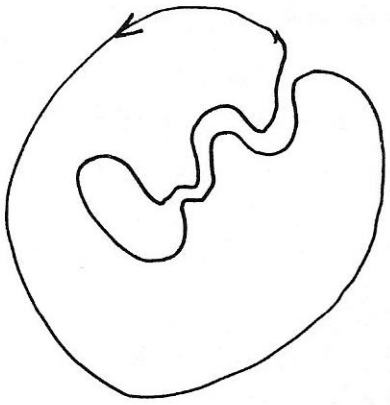


2. Find the possible winding numbers of the following curves around different points of the plane, and mark examples of each point on a sketch. [Hint - come in from 'infinity'.]

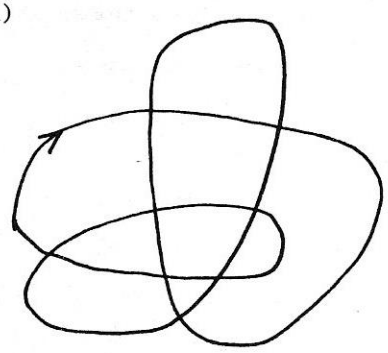
(i)



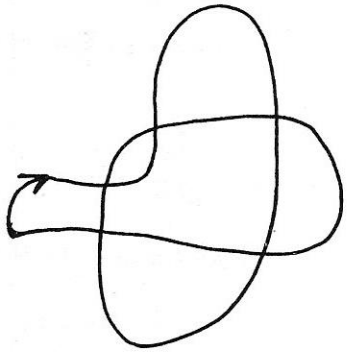
(ii)



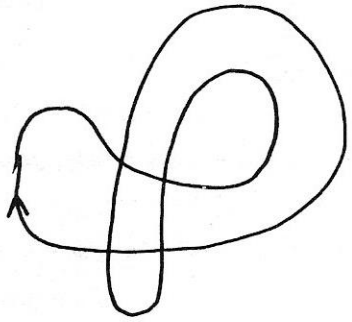
(iii)



(iv)



(v)



Section 7.2

When curves are given in parametric form careful sketching is often enough to enable one to calculate winding numbers.

Example

Find the winding number of

$$\left.\begin{aligned} x &= \sin t \\ y &= \cos 2t + \cos 4t \end{aligned}\right\} \quad 0 \leq t \leq 2\pi$$

about (0,0).

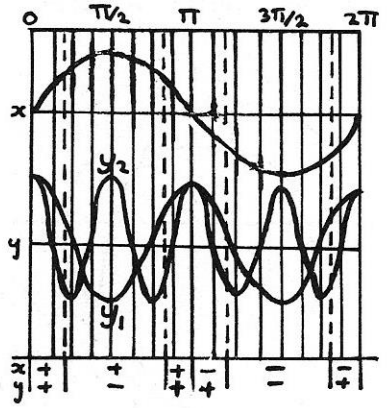
Observe that it is worthwhile knowing the points where sint, cos 2t, cos 4t are either ±1 (their extreme values) or zero, which will help us in determining the sign of x and y. These points are

(for sin t): $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

(for cos 2t): $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$

(for cos 4t): $t = \frac{k\pi}{8}$, $k = 0, 1, \ldots, 16$.

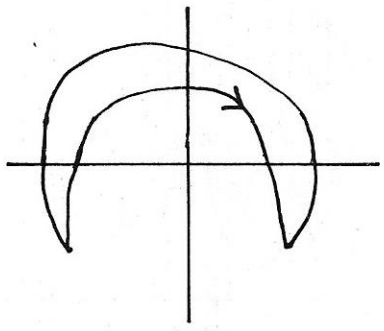
Plot these points on two horizontal axes, one for the graph of x, one for the graph of y, keeping them neatly aligned. Then draw the graphs of $x = \sin t$, $y_1 = \cos 2t$, $y_2 = \cos 4t$. You should get something like this:



There's no need to be accurate about the vertical heights, but it's worthwhile getting the curves to do what they ought to do at the points where $t = k\pi/8$.

Now, $y = y_1 + y_2$. There's plainly a point between $\frac{\pi}{8}$ and $\frac{\pi}{4}$ where y goes negative, so draw a vertical line (as shown by our dotted one). Do this all the way along, and you can read off the behaviour of x and y simultaneously. What happens at $t = \frac{\pi}{2}$? $y_1 = -1$, $y_1 = +1$, so $y = y_1 + y_2 = 0$. Since the slope of y_1 near $t = \frac{\pi}{2}$ is always less than or equal to the slope of y_2 in that range, $|y_2| \leq |y_1|$, so here $y_1 + y_2 \leq 0$. (You might literally see these inequalities in looking near $t = \pi$). In particular, the sign of y does not change at $t = \frac{\pi}{2}$. Notice that we haven't had to plot y itself: it's enough to know when it changes sign.

We calculate the winding number by drawing schematically any closed curve which travels between the quadrants in the same way as the given one does, namely like this:



The winding number is zero.

1. Find the winding number at (0,0) of each of the following curves

(i) $C_1: x = \cos t + \sin t$, $y = \sin 2t$ $\Big\}$ $0 \leq t \leq 2\pi$

(ii) $C_2: x = \cos t + \cos 3t$, $y = \sin t + \frac{1}{4}$ $\Big\}$ $0 \leq t \leq 2\pi$

(iii) $C_3: x = \cos t + \sin t + 3$, $y = \sin t$ $\Big\}$ $0 \leq t \leq 2\pi$

(iv) $C_4: x = \cos(t + \frac{\pi}{4})$, $y = \sin t$ $\Big\}$ $0 \leq t \leq 2\pi$

(v) $C_5: x = \cos t + \cos 3t$, $y = \sin 2t + \frac{1}{4}$ $\Big\}$ $0 \leq t \leq 2\pi$

[You may assume that y has four roots t_1, t_2, t_3, t_4, and $\frac{\pi}{2} < t_1 < \frac{2\pi}{3}$; $\frac{5\pi}{6} < t_2 < \pi$; $\frac{3\pi}{2} < t_3 < \frac{5\pi}{3}$; $\frac{11\pi}{6} < t_4 < 2\pi$.]

Section 7.3

Example

B page 132, Question 4 asks you to show that the zeros of $F(z) = z^6 + 2z^3 + 3z^2 + 5z + 15 = 0$ lie in the annulus $1 \leq |z| \leq 2$. To do this, consider first C_2, $|z| = 2$. Let $f(z) = 2z^3 + 3z^2 + 5z + 15$ and $g(z) = z^6$, so $F(z) = f(z) + g(z)$. On C_2,

$$|f(z)| \leq 2|z^3| + 3|z^2| + 5|z| + 15$$
$$= 2.8 + 3.4 + 5.2 + 15$$
$$= 53$$

and $|g(z)| = 2^6 = 64$, so $|f(z)| < |g(z)|$. Consequently $V_{C_2}(F) = V_{C_2}(f + g) = V_{C_2}(g) = 6.2\pi$, by Rouché's Theorem. So all the zeros of F lie inside $|z| = 2$. Now consider C_1, $|z| = 1$.

Let $f(z) = z^6 + 2z^3 + 3z^2 + 5z$, and $g(z) = 15$, so $F(z) = f(z) + g(z)$.

On C_1, $|f(z)| \leq |z^6| + 2|z^3| + 3|z^2| + 5|z|$

$= 1 + 2 + 3 + 5$

$= 11$

and $|g(z)| = 15$, so $|f(z)| < |g(z)|$ on C_1.

Consequently $V_{C_1}(F) = V_{C_1}(f + g) = V_{C_1}(g) = 0$, by Rouché's Theorem. So all the zeros of F lie outside $|z| = 1$, and hence in the annulus $1 \leq |z| \leq 2$, as claimed.

Although we shall not attempt to be much more precise about the location of the roots, we note some other techniques which can help.

(i) Since the coefficients of F are all real the non-real roots occur in complex conjugate pairs. So if $\alpha + i\beta$ is a root, then so is $\alpha - i\beta$. This means that every root in the first quadrant has a conjugate in the fourth quadrant, and every root in the second quadrant corresponds to one in the third and vice versa.

(ii) When looking for real roots, the derivative of F can be useful. For by confining z to the real axis and looking at $F'(x)$ $(x \in \mathbb{R})$ we can hope to use calculus arguments to detect zeros of F (or their absence).

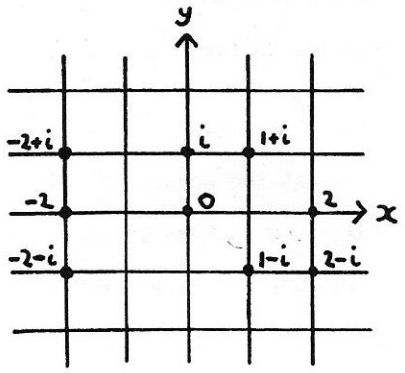
1. Show that the zeros of $F(z) = z^3 + z + 1$ lie between $|z| = \frac{1}{2}$ and $|z| = 2$; that one is negative and real and that the other two lie in the first and fourth quadrants.

2. Show that the zeros of $F(z) = z^3 - z + 4$ lie between $|z| = 1$ and $|z| = 2$, that one is negative and real and that the other two lie in first and fourth quadrants.

3. Show that $F(z) = z^4 + z + 4$ has all its roots in the annulus $A = \{z: 1 \leq |z| \leq 2\}$, and moreover that one lies in each quadrant.

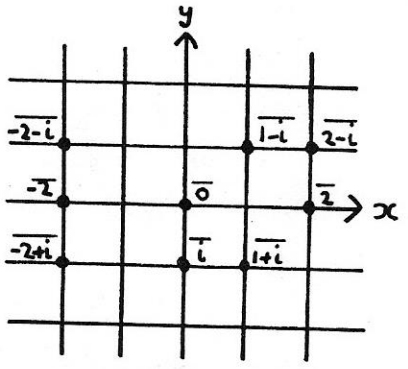
Section 5.1

1.



2. $(x+yi)\,\dfrac{(x-yi)}{x^2+y^2} = \dfrac{x^2+y^2}{x^2+y^2} = 1.$

3.



4. $z^{-1} = \dfrac{x-yi}{x^2+y^2}$ (by 2)

$= \bar{z}/|z|^2.$

Notice that $|z|^2 = z\bar{z}$.

5. $|2| = 2$, $|0| = 0$, $|i| = 1$, $|1+i| = \sqrt{2}$, $|1-i| = \sqrt{2}$, $|2-i| = \sqrt{5}$, $|-2| = 2$, $|-2-i| = \sqrt{5}$, $|-2+i| = \sqrt{5}$.

6. When n = 2,

$e^{i2\theta} = \cos 2\theta + i \sin 2\theta = (\cos\theta + i \sin\theta)^2$

$= \cos^2\theta - \sin^2\theta + 2i \cos\theta \sin\theta,$

so, equating real and imaginary parts,

$\cos 2\theta = \cos^2\theta - \sin^2\theta,$

$\sin 2\theta = 2 \cos\theta \sin\theta.$

When n = 3,

$e^{i3\theta} = \cos 3\theta + i \sin 3\theta = (\cos\theta + i \sin\theta)^3$

$= \cos^3\theta - 3 \cos\theta \sin^2\theta + 3i \cos^2\theta \sin\theta$

$-i\sin^3\theta,$

so equating real and imaginary parts,

$\cos 3\theta = \cos^3\theta - 3 \cos\theta \sin^2\theta$

$= 4 \cos^3\theta - 3 \cos\theta,$

$\sin 3\theta = 3 \cos^2\theta \sin\theta - \sin^3\theta$

$= 3 \sin\theta - 4 \sin^3\theta.$

There is a formula for any n, obtained by expanding $(\cos\theta + i \sin\theta)^n$ in the same way.

7. $z^{-1} = \bar{z}$ goes round the unit circle clockwise.

$\bar{z}^{-1} = z$ goes round it anticlockwise.

8. Let $a = a_1 + i\,a_2$, $b = b_1 + i\,b_2$, then

$|a+b| \leq |a| + |b| \iff$

$|a+b|^2 \leq (|a| + |b|)^2$ (since both sides are non-negative) $\iff$

$(a_1 + b_1)^2 + (a_2 + b_2)^2 \leq a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2|a||b| \iff$

$2(a_1 b_1 + a_2 b_2) \leq 2|a||b| \iff$

$(a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2) \iff$

$2a_1 b_1 a_2 b_2 \leq a_1^2 b_2^2 + a_2^2 b_1^2 \iff$

$0 \leq (a_1 b_2 - a_2 b_1)^2,$

which is true; so

$|a+b| \leq |a| + |b|.$

Replacing a by a - b yields

$|a| \leq |a-b| + |b|$, i.e.

$|a| - |b| \leq |a-b|.$

Replacing b by -b in the first inequality yields $|a - b| \leq |a| + |b|$, so $|a| - |b| \leq |a + b| \leq |a| + |b|$.

$|a + b|$ is nearly $|a|$, when $|b|$ is small.

9. 2, 3, 4, 2, 2, 3, 4, 5, 3, 3.

For each fixed z the equation is a polynomial in w which generally has as many distinct solutions as the highest power of w.

10. (i) (0,0) is a branch point of order 1 (= 2 - 1); see the text.

(ii) (0,0) is a branch point of order 2. The distinct w points corresponding to $z = re^{i\theta}$ for small r are $w = r^{1/3} e^{i(\theta/3 + 2k\pi/3)}$, $k = 0, 1, 2$. There are 3 of these, whereas if $z = 0$, $w^3 = 0$ has a unique solution. So (0,0) is a branch point of order 2.

(iii) (0,0) is a branch point of order 6 for the same reason.

(iv) $w^3 = z^2$ has a branch point of order 2 at (0,0) since generally three values $w = r^{2/3} e^{i2\theta/3}$, $r^{2/3} e^{i\,2(\theta+\pi)/3}$, $r^{2/3} e^{i2(\theta+2\pi)/3}$ correspond to $z = re^{i\theta}$, but there is only one w-value ($w = 0$) corresponding to $z = 0$.

(v) When z is nearly 0 in modulus, $z + 2$ is nearly 2. So as z circles $z = 0$ once, so does $z(z + 2)$, so the behaviour of $w^3 = z(z + 2)$ must be essentially the same as that of $w^3 = z$ near $z = 0$. Accordingly, $z = 0$ is a branch point of order 2.

11. (i) (a) Consider $z = 1/\tilde{z}$. $(z,w) = (\infty,w)$ lies on the locus if and only if $(\tilde{z},w) = (0,w)$ lies on the locus $w^2\tilde{z} + 2w + 2\tilde{z} = 0$, which (0,0) does. So $(z,w) = (\infty,0)$ lies on the locus. Since $2w = 0$ is linear, the point has multiplicity 1.

(b) Consider $w = 1/\tilde{w}$. $(z,w) = (z,\infty)$ lies on the locus if and only if $(z,\tilde{w}) = (z,0)$ lies on $1 + 2\tilde{w}z + 2\tilde{w}^2 = 0$, which is impossible. No point (z,∞) lies on the curve.

(c) Consider $z = 1/\tilde{z}$, $w = 1/\tilde{w}$. $(z,w) = (\infty,\infty)$ lies on the surface if and only if $(\tilde{z},\tilde{w}) = (0,0)$ lies on $\tilde{z} + 2\tilde{w} + 2\tilde{w}^2\tilde{z} = 0$ which it does. The point $z = \infty$ is covered by exactly as many points as any other z-point (counting with multiplicity). So, by (a), the multiplicity of (∞,∞) is 1. Alternatively, set $\tilde{z} = 0$; then $2\tilde{w} = 0$ is of degree 1, so

the solution $\tilde{w} = 0$ ($w = \infty$) occurs with multiplicity 1.

(ii) (a) Let $z = 1/\tilde{z}$.

$(z,w) = (\infty,w)$ lies on the locus if and only if $(\tilde{z},w) = (0,\infty)$ lies on the locus $3w^2\tilde{z}^2 + 2 + \tilde{z}^2 = 0$, which is impossible.

(b) Let $w = 1/\tilde{w}$.

$(z,w) = (z,\infty)$ lies on the locus if and only if $(z,\tilde{w}) = (z,0)$ lies on the locus $3 + 2z^2\tilde{w}^2 + \tilde{w}^2 = 0$, which is impossible.

(c) Let $z = 1/\tilde{z}$, $w = 1/\tilde{w}$.

(∞,∞) lies on the locus if and only if $(\tilde{z},\tilde{w}) = (0,0)$ lies on $3\tilde{z}^2 + 2\tilde{w}^2 + \tilde{w}^2\tilde{z}^2 = 0$.

When $\tilde{z} = 0$ this reduces to $2\tilde{w}^2 = 0$, which has $\tilde{w} = 0$ as a double root, so (∞,∞) lies on the original locus with multiplicity 2.

(iii) (a) Let $z = 1/\tilde{z}$.

$(z,w) = (\infty,w)$ lies on the locus if and only if $(\tilde{z},w) = (0,w)$ lies on the locus $w^2\tilde{z}^3 = -1$, which is impossible.

(b) Similarly there are no points (z,∞) on the locus.

(c) Let $z = 1/\tilde{z}$, $w = 1/\tilde{w}$.

$(z,w) = (\infty,\infty)$ lies on the locus if and only if $(\tilde{z},\tilde{w}) = (0,0)$ lies on $\tilde{z}^3 = -\tilde{w}^2$, which it does with multiplicity 2 (the degree of $0 = \tilde{w}^2$ as an equation in $\tilde{w}$). So (∞,∞) lies on the locus with multiplicity 2.

(iv) (a) Let $z = 1/\tilde{z}$.

$(z,w) = (\infty,w)$ lies on the locus if and only if $(\tilde{z},w) = (0,w)$ lies on the locus $w^4\tilde{z}^3 + 2w^2\tilde{z}^2 + 1 = 0$, which is impossible.

(b) Let $w = 1/\tilde{w}$.

$(z,w) = (z,\infty)$ lies on the locus if and only if $(z,\tilde{w}) = (z,0)$ lies on the locus $1 + 2\tilde{w}^2z + z^3\tilde{w}^4 = 0$, which is impossible.

(c) Let $z = 1/\tilde{z}$ and $w = 1/\tilde{w}$.

Then $(z,w) = (\infty,\infty)$ lies on the locus if and only if $(0,0)$ lies on the locus $\tilde{z}^3 + 2\tilde{w}^2\tilde{z}^2 + \tilde{w}^4 = 0$, which reduces to $\tilde{w}^4 = 0$, of which $\tilde{w} = 0$ is a 4-fold root. So $(z,w) = (\infty,\infty)$ is a 4-fold point on the original surface.

(v) (a) none

(b) none

(c) (∞,∞) is a triple point.

(vi) (a) Let $z = 1/\tilde{z}$.

$(z,w) = (\infty,w)$ lies on the

locus if and only if

$(\tilde{z},w) = (0,w)$ lies on

$w^{4}\tilde{z}^{2} + 3w\tilde{z} + w + 2\tilde{z}^{2} = 0$

which it does for $w = 0$.

(b) $(z,w) = (z,\infty)$ lies on the locus if and only if

$1 + 3\tilde{w}^{3}z + \tilde{w}^{3}z^{2} + 2\tilde{w}^{4} = 0$

which is impossible.

(c) (∞,∞) lies on the surface if and only if $(\tilde{z},\tilde{w}) = (0,0)$ lies on

$\tilde{z}^{2} + 3\tilde{w}^{3}\tilde{z} + \tilde{w}^{3} + 2\tilde{w}^{4}\tilde{z}^{2} = 0$

which it does with multiplicity 3, as it reduces to $\tilde{w}^{3} = 0$.

(vii) Carrying through the argument of (vi) we find $(\infty,0)$, $(0,\infty)$ each lie on the surface with multiplicity 1, and (∞,∞) twice.

12. (i) When $z = 0$ the equation reduces to $-w^{2} + 1 = 0$ so there are two corresponding finite (z,w) points on the locus, $(0,1)$ and $(0,-1)$.

When $z = 1$ the equation reduces to

$w^{3} - w^{2} - 2w + 2 = 0$, which factorizes as

$(w - 1)(w^{2} - 2) = 0$, i.e. as

$(w - 1)(w + \sqrt{2})(w - \sqrt{2}) = 0$,

so there are 3 corresponding (z,w) points on the locus: $(1,1)$, $(1, -\sqrt{2})$, $(1, \sqrt{2})$. All these points are finite.

When z takes a general value, there will be 3 corresponding points (z,w) on the locus, because the equation is of degree 3 in w.

(ii) When $z = 0$ the equation reduces to $-w^{2} - 2 = 0$, so there are two corresponding (z,w) points on the locus, $(0, i\sqrt{2})$ and $(0, -i\sqrt{2})$.

When $z = 1$ the equation reduces to $-w - 1 = 0$, so there is only one corresponding finite point on the locus, $(1, -1)$.

In general there are two points on the locus for each value of z because the equation is of degree 2 in w.

(iii) When $z = 0$ the equation reduces to $1 = 0$, which is false, so there are no corresponding finite points on the locus.

When $z = 1$ the equation reduces to $-w = 0$, so there is one corresponding finite point on the locus, $(1,0)$.

For a general value of z there will be two corresponding finite points on the locus, since the equation is of degree 2 in w.

(iv) When $z = 0$ the equation reduces to $-w^{3} = 0$, so there is only one corresponding finite point on the locus, $(0,0)$.

When $z = 1$ the equation reduces to $-w^{2} + 3 = 0$, so there are only two

corresponding points on the locus, $(1, \sqrt{3})$ and $(1, -\sqrt{3})$. In general there will be 3 finite points on the locus corresponding to a value of z, since the equation is of degree 3 in w.

(v) When $z = 0$ the equation reduces to $-2w^4 + 2 = 0$, so there are 4 corresponding finite points on the locus, $(0,1)$ $(0,i)$, $(0,-1)$, and $(0,-i)$.

When $z = 1$ the equation reduces to $-w^4 - w^2 + 2 = 0$, which can be factorized as $(w^2 + 2)(w^2 - 1) = 0$. So there are four corresponding finite points on the locus, $(1, i\sqrt{2})$, $(1, -i\sqrt{2})$, $(1,1)$ and $(1, -1)$.

In general there will be 4 values of w for each value of z, since the equation is of degree 4 in w.

(vi) When $z = 0$ the equation reduces to $w^4 = 0$ so there is only one corresponding finite point on the locus, $(0,0)$.

When $z = 1$ the equation reduces to $2w^4 - 1 = 0$, so there are 4 corresponding finite points on the locus; $(1, (\frac{1}{2})^{1/4})$, etc.

In general there will be 4 values of w for each value of z since the equation is of degree 4 in w.

Sections 5.2 and 5.3

1. The reductions are:

   (i) $(w + z + 1)(w + 2z - 1)$

   (ii) $(w + 2z - 1)(2w - z)$

   (iii) $(w + 1)(w - z - 2)$

   (iv) $(w - 2z)(w - z - 1)$

   (v) $(2z + 1)(w - 2)$

   These can either be spotted by trial and error, or by using the test in the unit.

2. (a) has no term in w^2, but is irreducible, so it represents a sphere homeomorphic to the z-plane.

   (b) similarly describes a sphere.

   (c) $w^2 - 2w - z(z + 1) = 0$

   $$\Rightarrow w = \frac{2 \pm \{4 + 4z(z + 1)\}^{1/2}}{2}$$

   $$= 1 \pm (1 + z + z^2)^{1/2}.$$

   There are branch points over those values of z for which $1 + z + z^2 = 0$ i.e. $z = -\frac{1}{2} \pm \frac{i}{2}\sqrt{3}$; so the branch points are

   $(-\frac{1}{2} + \frac{i}{2}\sqrt{3}, 1)$ and $(-\frac{1}{2} - \frac{i\sqrt{3}}{2}, 1)$.

   $(z,w) = (\infty,w)$, $w \neq \infty$, lies on the locus $\Leftrightarrow (\tilde{z},w) = (0,w)$ lies on the locus $\tilde{z}^2\tilde{w}^2 - 2w\tilde{z}^2 - 1 - \tilde{z} = 0$, which it does not.

   $(z,w) = (z,\infty)$, $z \neq \infty$, lies on the locus $\Leftrightarrow (z,\tilde{w}) = (z,0)$ lies on the locus

   $1 - 2\tilde{w} - \tilde{w}^2 z(z + 1) = 0$, which it does not.

$(z,w) = (\infty,\infty)$ lies on the locus

$\Leftrightarrow (\tilde{z},\tilde{w}) = (0,0)$ lies on the locus

$\tilde{z}^2 - 2\tilde{w}\tilde{z}^2 - \tilde{w}^2(1 + \tilde{z}) = 0$, which it does. It is double point, since $\tilde{w} = 0$ is a repeated zero of $\tilde{w}^2 = 0$. To see if it is a branch point or a pinch point we solve the tilde equation for $\tilde{w}$, and obtain

$$\tilde{w} = \frac{-2\tilde{z}^2 \pm [4\tilde{z}^4 + 4\tilde{z}^2(1 + \tilde{z})]^{1/2}}{2(1 + \tilde{z})}$$

$$= \frac{-2\tilde{z}^2 \pm 2\tilde{z}[\tilde{z}^2 + \tilde{z} + 1]^{1/2}}{2(1 + \tilde{z})}.$$

Since $\tilde{z}^2 + \tilde{z} + 1$ is not zero when $\tilde{z} = 0$, the point is not a branch point and must be a pinch point.

The locus is a pinched sphere which double covers the z-sphere and is branched over

$$z = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}.$$

(d) $w^2 + 2zw + z^2 - z + 3 = 0 \Rightarrow$

$$w = \frac{-2z \pm \{4z^2 - 4(z^2 - z + 3)\}^{1/2}}{2}$$

$$= -z \pm (z - 3)^{1/2}.$$

So $(z,w) = (3, -3)$ is a branch point.

$(z,w) = (\infty,w)$, $w \neq \infty$, lies on the locus $\Leftrightarrow$

$(\tilde{z},w) = (0,w)$ lies on the locus

$w^2\tilde{z}^2 + 2\tilde{z}w + 1 - \tilde{z} + 3\tilde{z}^2 = 0$, which it does not.

$(z,w) = (z,\infty)$, $z \neq \infty$, lies on the locus $\Leftrightarrow$

$(z,\tilde{w}) = (z,0)$ lies on

$1 + 2z\tilde{w} + (z^2 - z + 3)\tilde{w}^2 = 0$, which it does not.

$(z,w) = (\infty,\infty)$ lies on the locus

$\Leftrightarrow (\tilde{z},\tilde{w}) = (0,0)$ lies on

$\tilde{z}^2 + 2\tilde{z}\tilde{w} + (1 - \tilde{z} + 3\tilde{z}^2)\tilde{w}^2 = 0$,

which it does. Solving this equation for $\tilde{w}$, we find

$$\tilde{w} = \frac{-2\tilde{z} \pm [4\tilde{z}^2 - 4\tilde{z}^2(1 - \tilde{z} + 3\tilde{z}^2)]^{1/2}}{2(1 - \tilde{z} + 3\tilde{z}^2)}$$

$$= \frac{-2\tilde{z} \pm 2\tilde{z}[\tilde{z} - 3\tilde{z}^2]^{1/2}}{2(1 - \tilde{z} + 3\tilde{z}^2)}.$$

Since $\tilde{z} - 3\tilde{z}^2 = 0$ when $\tilde{z} = 0$, (∞,∞) is a branch point of the locus.

The locus is a sphere which double covers the z-sphere and is branched over $z = 3$ and $z = \infty$.

(e) $w^2 + 2(3z + 1)w - z + 3 = 0$

$$\Rightarrow w = \frac{1}{2}[-2(3z + 1) \pm (4(3z + 1)^2 + 4(z - 3))]^{1/2}$$

$$= -(3z + 1) \pm (9z^2 + 7z - 2)^{1/2}$$

$$= -(3z + 1) \pm \{(9z - 2)(z + 1)\}^{1/2}.$$

So the surface is branched over $z = 2/9$ and $z = -1$, i.e. $(z,w) = (2/9, -5/3)$ and $(-1,2)$ are branch points.

$(z,w) = (\infty,w)$, $w \neq \infty$, lies on the locus $\Leftrightarrow$ $(\tilde{z},w) = (0,w)$ lies on the locus

$6w + w^2\tilde{z} - 1 + (2w + 3)\tilde{z} = 0$, which it does when $w = 1/6$.

So $(z,w) = (\infty, 1/6)$ lies on the locus.

$(z,w) = (z,\infty)$ lies on the locus

$\Leftrightarrow (z,\tilde{w}) = (z,0)$ lies on the locus

$6z\tilde{w} + 1 - z\tilde{w}^2 + 2\tilde{w} + 3\tilde{w}^2 = 0$, which it does not.

$(z,w) = (\infty,\infty)$ lies on the locus $\Leftrightarrow$ $(\tilde{z},\tilde{w}) = (0,0)$ lies on the locus $6\tilde{w} + \tilde{z} - \tilde{w}^2 + (2\tilde{w} + 3\tilde{w}^2)\tilde{z} = 0$, which it does.

The locus is a sphere which double covers the z-sphere and is branched over $z = 2/9$ and $z = -1$.

(f) $w^2 + 4zw - 4(z - 1) = 0$

$$\Rightarrow w = \frac{-4z \pm \{16z^2 + 16(z - 1)\}^{1/2}}{2}$$

$$= -2z \pm 2\{z^2 + z - 1\}^{1/2}.$$

So the surface is branched over the points z for which $z^2 + z - 1 = 0$, i.e. over $z = \frac{-1 \pm \sqrt{5}}{2}$, i.e. at $(z,w) = \left(\frac{-1 + \sqrt{5}}{2},\ 1 - \sqrt{5}\right)$ and $\left(\frac{-1 - \sqrt{5}}{2},\ 1 + \sqrt{5}\right)$.

$(z,w) = (\infty,w)$, $w \neq \infty$, lies on the locus $\Leftrightarrow$ $(\tilde{z},w) = (0,w)$ lies on the locus $w^2\tilde{z} + 4w - 4(1 - \tilde{z}) = 0$, which it does when $w = 1$.

So $(z,w) = (\infty,1)$ lies on the locus.

$(z,w) = (z,\infty)$, $z \neq \infty$, lies on the locus $\Leftrightarrow$ $(z,\tilde{w}) = (z,0)$ lies on the locus $1 + 4z\tilde{w} - 4(z - 1)\tilde{w}^2 = 0$, which it does not.

$(z,w) = (\infty,\infty)$ lies on the locus $\Leftrightarrow$ $(\tilde{z},\tilde{w}) = (0,0)$ lies on the locus $\tilde{z} + 4\tilde{w} - 4\tilde{w}^2(1 - \tilde{z}) = 0$, which it does. But (∞,∞) is an ordinary point since the roots of $4\tilde{w} - 4\tilde{w}^2 = 0$ are $\tilde{w} = 0$ and 1.

The locus is a sphere which double covers the z-sphere and is branched over the two points mentioned above.

(g) $w^2 - zw - (2z^2 + 3) = 0$

$$\Rightarrow w = \frac{1}{2}[z \pm \{z^2 + 4(2z^2 + 3)\}^{1/2}]$$

$$= \frac{1}{2}[z \pm (9z^2 + 12)^{1/2}],$$

So the surface is branched over $\pm i\sqrt{\frac{4}{3}} = \pm \frac{2i}{\sqrt{3}}$, i.e. at $\left(\frac{2i}{\sqrt{3}},\ \frac{i}{\sqrt{3}}\right)$ and $\left(\frac{-2i}{\sqrt{3}},\ \frac{-i}{\sqrt{3}}\right)$.

$(z,w) = (\infty,w)$, $w \neq \infty$, lies on the locus $\Leftrightarrow$ $(\tilde{z},w) = (0,w)$ lies on $2 + \tilde{z}w - \tilde{z}^2(w^2 - 3) = 0$ which it does not.

$(z,w) = (z,\infty)$, $z \neq \infty$, lies on the locus $\Leftrightarrow$ $(z,\tilde{w}) = (z,0)$ lies on $2z^2\tilde{w}^2 + \tilde{w}z - 1 + 3\tilde{w}^2 = 0$, which it does not.

$(z,w) = (\infty,\infty)$ lies on the locus $\Leftrightarrow$ $(\tilde{z},\tilde{w}) = (0,0)$ lies on the locus $\tilde{z}^2 - \tilde{z}\tilde{w} - \tilde{w}^2(2 + 3\tilde{z}^2) = 0$, which it does, twice.

It is a pinch point, since the w-values of points near $(\tilde{z},\tilde{w}) = (0,0)$ are given by

$$\tilde{w} = \frac{-\tilde{z} \pm [\tilde{z}^2 + 4\tilde{z}^2(2 + 3\tilde{z}^2)]^{1/2}}{2(2 + 3\tilde{z}^2)}$$

$$= \frac{-\tilde{z} \pm \tilde{z}[9 + 12\tilde{z}^2]^{1/2}}{2(2 + 3\tilde{z}^2)}$$

and $9 + 12\tilde{z}^2$ does not vanish when $\tilde{z} = 0$.

(h) The coefficient of w^2 is zero, so the surface is homeomorphic to the z-sphere.

(i) $w^2 + (2z + 1)w + z^2 + 2 = 0 \Rightarrow$

$$w = \frac{-(2z + 1) \pm \{(2z + 1)^2 - 4(z^2 + 2)\}^{1/2}}{2}$$

$$= -z - 1/2 \pm \frac{1}{2}[4z - 7]^{1/2}$$

$$= -z - 1/2 \pm [z - \tfrac{7}{4}]^{1/2}.$$

So the surface has a branch point at $(\frac{7}{4}, -\frac{9}{4})$.

$(z,w) = (\infty,w)$, $w \neq \infty$, lies on the locus $\Leftrightarrow$ $(\tilde{z},w) = (0,w)$ lies on $w^{2\sim 2} + (2\tilde{z} + \tilde{z}^2)w + 1 + 2\tilde{z}^2 = 0$, which it does not.

$(z,w) = (z,\infty)$, $z \neq \infty$, lies on the locus $\Leftrightarrow$ $(z,\tilde{w}) = (z,0)$ lies on the locus $1 + (2z + 1)\tilde{w} + (z^2 + 2)\tilde{w}^2 = 0$, which it does not.

$(z,w) = (\infty,\infty)$ lies on the locus $\Leftrightarrow$ $(\tilde{z},\tilde{w}) = (0,0)$ lies on the locus $\tilde{z}^2 + (2 + \tilde{z})\tilde{z}\tilde{w} + (1 + 2\tilde{z}^2)\tilde{w}^2 = 0$, which it does as a repeated point. In fact, $(z,w) = (\infty,\infty)$ is a branch point, since points near there have $\tilde{w}$ values given by

$$\tilde{w} = \frac{-(2+\tilde{z})\tilde{z} \pm \{(2+\tilde{z})^2\tilde{z}^2 - 4\tilde{z}^2(1+2\tilde{z}^2)\}^{1/2}}{2(1 + 2\tilde{z}^2)}$$

$$= \frac{-(2 + \tilde{z})\tilde{z} \pm \tilde{z}\{4\tilde{z} - 7\tilde{z}^2\}^{1/2}}{2(1 + 2\tilde{z}^2)},$$

and $4\tilde{z} - 7\tilde{z}^2$ vanishes when $\tilde{z} = 0$. So the locus is homeomorphic to a sphere which double covers the z-sphere and is branched at $(\frac{4}{7}, -\frac{4}{9})$ and (∞,∞).

(j) $w^2 + zw + z = 0 \Rightarrow$

$$w = \frac{-z \pm [z^2 - 4z]^{1/2}}{2}.$$

So the surface is branched at $(0,0)$ and $(4, -2)$. $(z,w) = (\infty,w)$, $w \neq \infty$, lies on the locus $\Leftrightarrow$ $(\tilde{z},w) = (0,w)$ lies on the locus $w^{2\sim}\tilde{z} + w + 1 = 0$, so $(\infty,-1)$ lies on the locus. $(z,w) = (z,\infty)$, $z \neq \infty$, lies on the locus $\Leftrightarrow$ $(z,\tilde{w}) = (z,0)$ lies on the locus $1 + z\tilde{w} + z\tilde{w}^2 = 0$, which is false. $(z,w) = (\infty,\infty)$ lies on the locus $\Leftrightarrow$ $(\tilde{z},\tilde{w}) = (0,0)$ lies on $\tilde{z} + \tilde{w} + \tilde{w}^2 = 0$, which it does.

So the locus is homeomorphic to the z-sphere, which it double covers, and it is branched at $(0,0)$ and $(4, -2)$.

(k) $w^2 - 3zw + z^2 - 4 = 0 \Rightarrow$

$$w = \frac{3z \pm [9z^2 - 4(z^2 - 4)]^{1/2}}{2}$$

$$= \frac{3z}{2} \pm \frac{1}{2}[5z^2 + 16]^{1/2}.$$

So the surface is branched at $\left(\frac{4i}{\sqrt{5}}, \frac{6i}{\sqrt{5}}\right)$, $\left(\frac{-4i}{\sqrt{5}}, \frac{-6i}{\sqrt{5}}\right)$. $(z,w) = (\infty,w)$, $w \neq \infty$, lies on the locus $\Leftrightarrow$ $(\tilde{z},w) = (0,w)$ lies on the locus $w^{2\sim 2} - 3\tilde{z}w + 1 - 4\tilde{z}^2 = 0$, which is false. $(z,w) = (z,\infty)$, $z \neq \infty$, lies on the locus $\Leftrightarrow$ $(z,\tilde{w}) = (z,0)$ lies on the locus $1 - 3z\tilde{w} + (z^2 - 4)\tilde{w}^2 = 0$, which is false. $(z,w) = (\infty,\infty)$ lies on the locus $\Leftrightarrow$ $(z,w) = (0,0)$ lies on the locus $\tilde{z}^2 - 3\tilde{z}\tilde{w} + (1 - 4\tilde{z}^2)\tilde{w}^2 = 0$, which it does as a repeated point. It is a pinch point, since, solving for $\tilde{w}$, we find

$$\tilde{w} = \frac{3\tilde{z} \pm \{9\tilde{z}^2 - 4\tilde{z}^2(1 - 4\tilde{z}^2)\}^{1/2}}{2(1 - 4\tilde{z}^2)}$$

$$= \frac{3\tilde{z} \pm \tilde{z}[5 + 16\tilde{z}^2]^{1/2}}{2(1 - 4\tilde{z}^2)}$$

and $5 + 16\tilde{z}^2$ does not vanish when $\tilde{z} = 0$. So the locus is homeomorphic to a pinched sphere. The corresponding unpinched sphere is a double cover of the z-sphere and is branched at $\left(\frac{4i}{\sqrt{5}}, \frac{6i}{\sqrt{5}}\right)$ and $\left(\frac{-4i}{\sqrt{5}}, \frac{-6i}{\sqrt{5}}\right)$. Notice that our conclusions all confirm the main theorem, as follows.

(a),(b) $C = 0$; locus is homeomorphic to the z-sphere.

(c) $A \neq 0$, $C \neq 0$, $B^2 - 4AC \neq 0$; general case (pinched double cover).

(d) $B^2 - 4AC = 0$; double cover, branched over $z = \infty$.

(e),(f) $A = 0$; double cover, $z = \infty$ an ordinary point of the z-sphere.

(g) $A \neq 0$, $C \neq 0$, $B^2 - 4AC \neq 0$; general case, as (c).

(h) $C = 0$; as (a).

(i) $B^2 - 4AC = 0$, as (d).

(j) $A = 0$; as (e).

(k) $A \neq 0$, $C \neq 0$, $B^2 - 4AC \neq 0$ general case, as (c).

Notice also that we never have a double cover branched over one or three points. Once this has been proved to be impossible, we can automatically deduce that

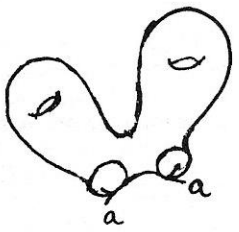
(i) if there is only one branch point over a finite value z then the conic is also branched over $z = \infty$, and

(ii) if there are two branch points over finite values of z, and the surface has a double point over $z = \infty$, then that point is a pinch point.

SOLUTIONS FOR UNIT 6

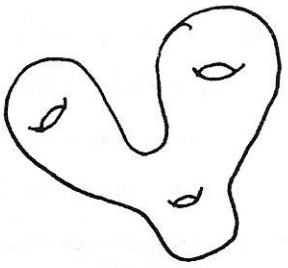
Section 6.1

1. S is almost

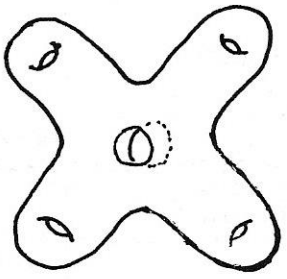


but the two edges a must be identified.

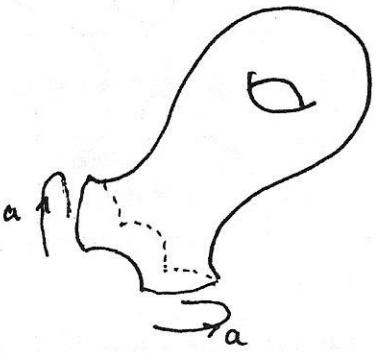
The resulting surface is a 3-holed torus.



2. $\tilde{S}$ is

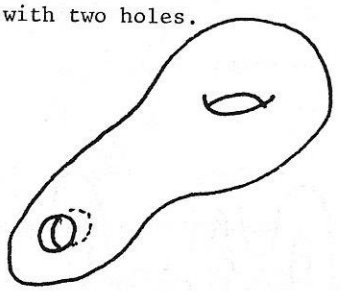


S is almost

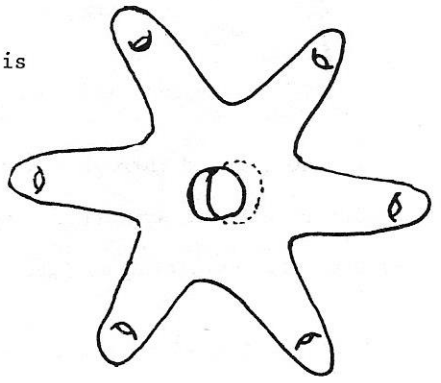


but the two edges must be identified.

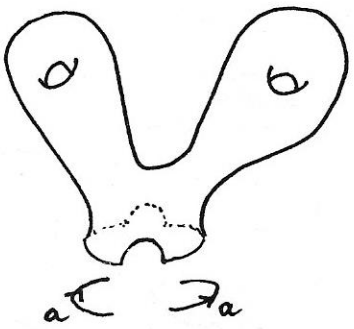
The resulting surface is a (one-holed) torus with two holes.



3. $\tilde{S}$ is

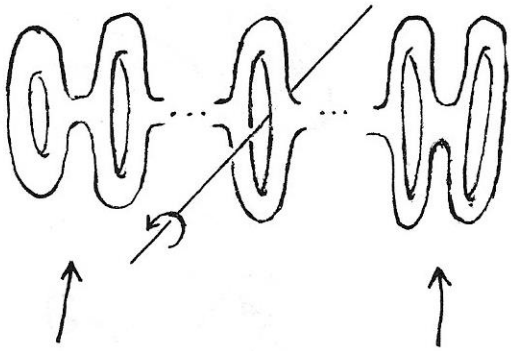


and we must consider the effects of rotations through $2\pi/3$ around the axis of symmetry. The resulting S is almost

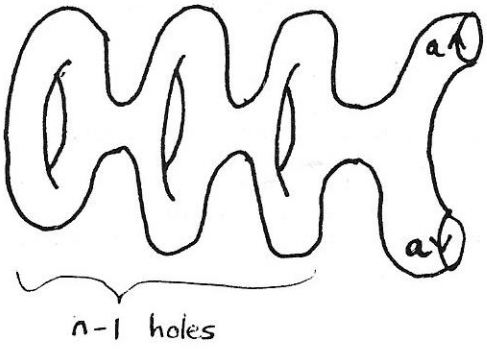


but the edges a must be identified. When this is done S is seen to be a two-holed torus with two holes.

4. Consider $\tilde{S}$, a (2n - 1)-holed torus, in this form:



It can be rotated through π about the indicated axis of symmetry. When this is done the resulting surface is almost



but the edges a must be identified. When this is done an n-holed torus is produced which is double covered by $\tilde{S}$.

Section 6.2

1. Let $\tilde{S}$ be an n-fold covering of the sphere with total branch point order δ (possibly zero). Then $\chi(\tilde{S}) = n\chi(S^2) - \delta$. So $\delta = n\chi(S^2) - \chi(\tilde{S}) \geq 2n - 2 = 2(n - 1)$. $(n - 1)$ is the largest deficit a branch point of an n-fold covering can have, so the covering must have at least 2 branch points.

2. Let $\tilde{S}$ be a two-fold covering of the sphere, then $\chi(\tilde{S}) = 2.\chi(S^2) - \delta$. But since $\tilde{S} = S^2$, $\chi(\tilde{S}) = 2$, so $\delta = 2.2 - 2 = 2$. The only possible branch point order for a double covering is 1, so there must be exactly two branch points.

3. Let $\tilde{S}$ be an n-fold covering of the sphere. Since the sphere is orientable, $\tilde{S}$ must be orientable. Since the sphere has no boundary, neither has $\tilde{S}$, so $\chi(\tilde{S})$ is even. But

$$\chi(\tilde{S}) = n.\chi(S^2) - \delta$$

so $\delta = n\chi(S^2) - \chi(\tilde{S})$;

and so δ, being the difference of two even numbers, must be even.

4. Let $\tilde{S}$ be a k-fold covering of S (a Riemann surface) with total branch point order δ. The same argument as in (3) shows that δ must be even.

5. The techniques are to check Euler characteristics, number of boundary components, and orientability.

(a) $\chi(T^2) = 0 \neq 3.\chi(S^2) - 0 = 6$, so this covering is impossible.

(b) $\chi(S^2) = 2 \neq 3.\chi(T^2) - 0 = 0$ so this covering is impossible.

(c) $2K^2$ is non-orientable but T^2 is, and a non-orientable surface cannot cover an orientable one.

(d) $\chi(4T^2) = -6 \neq 2.\chi(2T^2) - 0 = -4$ so this covering is impossible.

(e) is impossible for the same reason as (c).

(f) $\chi(4T^2) = -6 \neq 2.\chi(3K^2) - \delta = -8 - \delta$ so this covering is impossible.

(g) $\chi(5T^2) = -8 \neq 3.\chi(3K^2) - \delta = -12 - \delta$ so this covering is impossible.

(h) $\chi(3D^2) = -1 \neq 3.\chi(2D^2) - 0 = 0$ so this covering is impossible.

(i) $\chi(4D^2) = -2 \neq 3.\chi(3D^2) - \delta = -3 - \delta$ so this covering is impossible.

(j) $3M^2$ is non-orientable, and so cannot cover $2D^2$, which is orientable.

(k) $\chi(4T^2) = -6 \neq 3\chi(2T^2) - \delta = -6 - \delta$ when $\delta \geq 1$, so there cannot be such a branched covering.

(l) $\chi(9T^2) = -16 \neq 2.\chi(5T^2) - \delta = -16 - \delta$ when $\delta \geq 1$, so there cannot be such a branched covering.

(m) $\beta(7D^2) = 7 > 3\beta(K^2 \# 2D^2) = 6$, so it is false that $\beta(K^2 \# 2D^2) \leq \beta(7D^2) \leq 3\beta(K^2 \# 2D^2)$. So there cannot be such a covering.

6. $\chi(\tilde{S}) = \chi(\tilde{k}T^2 \# \tilde{b}D^2) = 2 - 2\tilde{k} - \tilde{b}$,

$\chi(S) = \chi(kT^2 \# bD^2) = 2 - 2k - b$.

So $\chi(\tilde{S}) = n\chi(S) - \delta \Rightarrow$

$2 - 2\tilde{k} - \tilde{b} = n(2 - 2k - b) - \delta$.

Moreover $b \leq \tilde{b} \leq nb$, so

$\delta = 2n - 2 + 2\tilde{k} - 2nk + \tilde{b} - nb$

$\leq 2\tilde{k} - 2 - 2nk + 2n$

$\leq 2\tilde{k} - 2$ since $k \geq 1$.

Section 6.3

(i) On inserting vertices, we find

$Pa_1\, Pb_1 P a_2^{-1} P b_2\, P = 1$

$P a_2 P b_2\, a_1^{-1} P b_3 P = 1$

$P a_3 P b_3\, Pa_3^{-1}\, Pb_1 P = 1$

so $\delta = 3 - 1 = 2$.

$\chi(S_1) = 3\chi(K^2) - 2 = -2$.

Since K^2 is without boundary, so is S_1. You can easily check that S_1 is non-orientable. Since $\beta(S_1) = 0$, the classification theorem tells us that $S_1 = 4\mathbb{RP}^2$.

(ii) Inserting vertices yields

$Pa_1\, Qb_1\, Ra_2^{-1}\, Rb_2\, P = 1$

$Ra_2\, Rb_2\, Pa_3^{-1}\, Qb_1\, R = 1$

$Qa_3\, Pb_3\, Qa_1^{-1}\, Pb_3\, Q = 1$;

$\delta = 3 - 3 = 0$.

$\chi(S^2) = 3.\chi(K^2) - 0 = 0$.

S_2 is non-orientable since it cannot be written so that each b_i occurs as $\ldots b_i \ldots b_i \ldots$.

$\beta(K^2) = 0 \Rightarrow \beta(S_2) = 0$. So the classification theorem tells us that $S_2 = K^2$.

1. (i) Inserting vertices yields

$Pa_1\ Pb_1\ Ra_2^{-1}\ Sb_4\ P = 1$

$Sa_2\ Rb_2\ Ta_3^{-1}\ Tb_3\ S = 1$

$Ta_3\ Tb_3\ Sa_4^{-1}\ Rb_2\ T = 1$

$Ra_4\ Sb_4\ Pa_1^{-1}\ Pb_1\ R = 1.$

So $\delta = 4 - 4 = 0$.

$\chi(S_2) = 4.0 - 0 = 0$, S_2 is clearly non-orientable, and

$\beta(K^2) = 0 \Rightarrow \beta(S_2) = 0$. So the classification theorem tells us that $S_3 \cong K^2$.

(ii) Inserting vertices yields

$Pa_1\ Pb_1\ Qa_2^{-1}\ Qb_2\ P = 1$

$Qa_2\ Qb_2\ Pa_1^{-1}\ Pb_3\ Q = 1$

$Pa_3\ Pb_3\ Qa_4^{-1}\ Qb_4\ P = 1$

$Qa_4\ Qb_4\ Pa_3^{-1}\ Pb_1\ Q = 1;$

$\delta = 4 - 2 = 2.$

$\chi(S_4) = \chi(K^2) - 2 = -2.$

$\beta(K^2) = 0 \Rightarrow \beta(S_4) = 0.$

S_4 is non-orientable because at least one b must occur as ...b...b... .

So the classification theorem says that $S_4 \cong 4\mathbb{RP}^2 \cong 2K^2$.

(iii) Inserting vertices yields that the vertex of K^2 is covered by only two vertices in S_5, so $\delta = 4 - 2 = 2$.

$\chi(S_5) = 4\chi(K^2) - 2 = -2.$

$\beta(S_5) = 0$ since $\beta(K^2) = 0$, and S_5 is non-orientable from the occurrence of $b_1 \ldots b_1$ in the first equation. So the classification theorem says that

$S_5 \cong 4\mathbb{RP}^2$.

2. (i) Inserting vertices yields

$Ra_1\ Ra_2\ Rb_1\ Uc_1\ Vb_2^{-1}\ R = 1$

$Ra_2\ Ra_1\ Rb_2\ Vc_2\ Ub_1^{-1}\ R = 1$

which, by comparison with the base surface

$Pa\ Pa\ Pb\ Qc\ Qb^{-1}\ P = 1,$

shows that the covering is branched over P but not over Q, so $\delta = 1$.

$\chi(S_6) = 2\chi(M^2) - 1 = -1.$

$\beta(S_6) = 1$, since the boundary consists of $Uc_1\ Vc_2\ U$.

The surface is non-orientable, since eliminating b_1 yields this edge equation:

$a_1\ a_2\ a_2\ a_1\ b_2\ c_2\ c_1\ b_2^{-1} = 1$

in which a_1 is repeated in the form $\ldots a_1 \ldots\ a_1 \ldots$.

So $S_6 \cong 2\mathbb{RP}^2\ \#\ D^2$ by the classification theorem.

(ii) Inserting vertices yields

$Ra_1\ Ta_2\ Rb_1\ Sc_1\ Sb_1^{-1}\ R = 1$

$Ta_2\ Ra_1\ Tb_2\ Uc_2\ Ub_2^{-1}\ T = 1,$

so the covering is unbranched, $\delta = 0$, and

$\chi(S_7) = 2\chi(M^2) - 0 = 0.$ $\beta(S_7) = 2$, since the boundary curves are $Sc_1 S$, $Uc_2 U$. It is orientable, since eliminating a yields the edge equation

$a_2\ b_1\ c_1^{-1}\ b_1^{-1}\ b_2\ c_2\ b_2^{-1} = 1$

in which every repeated edge occurs in the form ... X.. X... .

So $S_7 \cong aD^2$, the annulus, by the classification theorem.

3. (i) Inserting vertices yields

$Pc_1\ Qa_1\ Rf_1\ Sa_2\ Tb_1\ Ud_1\ Ve_1$
$Wd_2^{-1}\ P = 1;$

$Xc_2\ Sa_2\ Tf_2\ Za_3\ Ob_2\ Pd_2\ We_2$
$Yd_3^{-1}\ X = 1;$

$Uc_3\ Za_3\ Of_3\ Qa_1\ Rb_3\ Xd_3\ Ye_3$
$Vd_1^{-1}\ U = 1;$

which by comparison with

$Ac\ Ba\ Cf\ Ba\ Cb\ Ad\ Ee\ Fd^{-1}A = 1$

shows that each vertex in the base is covered by 3 distinct vertices, so the covering is unbranched; $\delta = 0$.

$\chi(S_8) = 3\,\chi(S) - 0 = -3.$

$\beta(S_8) = 4$, since the boundary components are

$Pc_1\ Qf_3^{-1}\ Ob_2\ P,$

$Xc_2\ Sf_1^{-1}\ Rb_3\ X,$

$Uc_3\ Zf_2^{-1}\ Tb_1\ U,$

$Ve_1\ We_2\ Ye_3\ V.$

Since $\chi(S_8) - \beta(S_8)$ is odd, S_8 is non-orientable and so, by the classification theorem,

$S_8 \cong \mathbb{RP}^2 \# 4D^2.$

(ii) Inserting vertices yields

$Pc_1\ Qa_1\ Rf_1\ Sa_2\ Tb_1\ Pd_1\ Ue_1$
$Ud_1^{-1}\ P = 1;$

$Zc_2\ Sa_2\ Tf_2\ Va_3\ Wb_2\ Zd_2\ Xe_2$
$Xd_2^{-1}\ Z = 1;$

$Nc_3\ Va_3\ Wf_3\ Qa_1\ Rb_3\ Nd_3\ Ye_3$
$Yd_3^{-1}\ N = 1.$

So the covering is unbranched, $\delta = 0$.

$\chi(S_9) = 3\chi(S) - 0 = -3.$

The boundary components are

Ue_1U, Xe_2X, Ye_3Y, and

$Pc_1\ Qf_3^{-1}\ Wb_2\ Zc_2\ Sf_1^{-1}\ Rb_3\ Nc_3$
$Vf_2^{-1}\ Tb_1\ P$ so $\beta(S_9) = 4$, and so as in 3(i), $S_9 \cong \mathbb{RP}^2 \# 4D^2$.

4. (i) $w^5 = z(z - \alpha)$.

We make a cut a from $P(z = \alpha)$ to $Q(z = 0)$ and b from Q to $R(z = \infty)$.

We are seeking a five-fold cover of the z-sphere (as described by $PaQbRb^{-1}Qa^{-1}\ P = 1$).

So we start with the system

$a_1\ b_1\ b_1^{-1}\ a_1^{-1} = 1;$
$a_2\ b_2\ b_2^{-1}\ a_2^{-1} = 1;$
$a_3\ b_3\ b_3^{-1}\ a_3^{-1} = 1;$
$a_4\ b_4\ b_4^{-1}\ a_4^{-1} = 1;$
$a_5\ b_5\ b_5^{-1}\ a_5^{-1} = 1;$

and investigate the interleaving over P, Q and R.

Consider a small circle around P. The five values of w over an ordinary point z may be written as

$w_1 = \sqrt[5]{z}\ \sqrt[5]{z - \alpha}\quad (-\frac{\pi}{5} < \operatorname{Arg} w_1 \leqq \frac{\pi}{5})$

$w_2 = e^{2\pi i/5}.w_1$

$w_3 = e^{4\pi i/5}.w_1$

$w_4 = e^{6\pi i/5}.w_1$

$w_5 = e^{8\pi i/5}.w_1$.

Moving around P, as we cross the edge a the values of w are

permuted by

$(w_1\ w_2\ w_3\ w_4\ w_5)$.

Thus one of the permutations to be applied to the above system is

$(a_1\ a_2\ a_3\ a_4\ a_5)$.

Next consider a loop enclosing P and Q. This crosses edge b, and since the loop encloses both P and Q the values of $\sqrt[5]{z}$ and $\sqrt[5]{z-\alpha}$ both change. Thus the required permutation is

$(b_1\ b_3\ b_5\ b_2\ b_4)$.

Applying this permutation to the above system we get

$a_1 b_1 b_3^{-1} a_2^{-1} = 1;$

$a_2 b_2 b_4^{-1} a_3^{-1} = 1;$

$a_3 b_3 b_5^{-1} a_4^{-1} = 1;$

$a_4 b_4 b_1^{-1} a_5^{-1} = 1;$

$a_5 b_5 b_2^{-1} a_1^{-1} = 1.$

Inserting vertices gives

$P_1 a_1 Q_1 b_1 R_1 b_3^{-1} Q_1 a_2^{-1} P_1 = 1;$

$P_1 a_2 Q_1 b_2 R_1 b_4^{-1} Q_1 a_3^{-1} P_1 = 1;$

$P_1 a_3 Q_1 b_3 R_1 b_5^{-1} Q_1 a_4^{-1} P_1 = 1;$

$P_1 a_4 Q_1 b_4 R_1 b_1^{-1} Q_1 a_5^{-1} P_1 = 1;$

$P_1 a_5 Q_1 b_5 R_1 b_2^{-1} Q_1 a_1^{-1} P_1 = 1.$

So we see that P, Q and R are each covered by only a single vertex. Each is a branch point with deficit 4.

Thus $\chi(\mathcal{S}) = 5.\chi(S^2) - 3.4$

$= 10 - 12$

$= -2.$

$\mathcal{S}$ (being closed and orientable) is a double torus.

(ii) Making cuts a from $P(z = \beta)$ to $Q(z = \alpha)$, b from Q to $R(z = 0)$ and c from R to $S(z = \infty)$, we seek four-fold covers of the sphere

$$PaQbRcSc^{-1}Rb^{-1}Qa^{-1}P = 1.$$

Write the four possible values of w as

$w_1 \qquad (-\frac{\pi}{4} < \text{Arg } w_1 \leq \frac{\pi}{4})$

$w_2 = e^{2\pi i/4}.w_1$

$w_3 = e^{4\pi i/4}.w_1$

$w_4 = e^{6\pi i/4}.w_1$.

Now consider a small circle around P. As z crosses a the effect on the w-values is $(w_1\ w_2\ w_3\ w_4)$, giving the corresponding permutation $(a_1\ a_2\ a_3\ a_4)$.

Next, with a loop enclosing P and Q, we find that as z crosses b the permutation is

$w_1 \mapsto e^{2\pi i/4}.e^{2\pi i/4}.w_1 = w_3$, and so on. The net effect on the b_i's is $(b_1\ b_3)(b_2\ b_4)$.

Finally, a loop around P, Q and R shows that as z crosses c the permutation is

$(c_1\ c_4\ c_3\ c_2)$.

Thus the system of edge equation for $\mathcal{S}$ is

$a_1 b_1 c_1 c_4^{-1} b_3^{-1} a_2^{-1} = 1;$

$a_2 b_2 c_2 c_1^{-1} b_4^{-1} a_3^{-1} = 1;$

$a_3 b_3 c_3 c_2^{-1} b_1^{-1} a_4^{-1} = 1;$

$a_4 b_4 c_4 c_3^{-1} b_2^{-1} a_1^{-1} = 1.$

Inserting vertices gives

$P_1a_1Q_1b_1R_1c_1S_1c_4^{-1}R_1b_3^{-1}Q_1a_2^{-1}P_1 = 1;$

$P_1a_2Q_1b_2R_1c_2S_1c_1^{-1}\ b_4^{-1}Q_1a_3^{-1}P_1 = 1;$

$P_1a_3Q_1b_3R_1c_3S_1c_2^{-1}R_1b_1^{-1}Q_1a_4^{-1}P_1 = 1;$

$P_1a_4Q_1b_4R_1c_4S_1c_3^{-1}R_1b_2^{-1}Q_1a_1^{-1}P_1 = 1.$

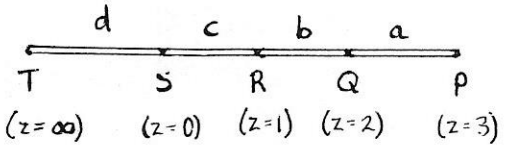
Thus P, Q, R and S are each covered by only one vertex and are thus branch points of deficit $4 - 1 = 3$.

So $\chi(\tilde{S}) = 4.\chi(S^2) - 4.3$

$= 8 - 12 = -4.$

As always for a Riemann surface, $\tilde{S}$ is closed and orientable and hence is a three-holed torus.

(iii) Let P be z = 3, Q be z = 2, R be z = 1, S be z = 0 and T be $z = \infty$. As usual we make cuts between these points.



We seek a 3-fold cover of the sphere.

$PaQbRcSdTd^{-1}Sc^{-1}Rb^{-1}Qa^{-1}P$

The usual method of taking loops around P; P & Q; P, Q & R; P, Q, R & S, yields the following permutations:

$(a_1\ a_2\ a_3)$

$(b_1\ b_3\ b_2)$

identity

$(d_1\ d_2\ d_3).$

The resulting system of edge equations (with vertices already inserted) is

$P_1a_1Q_1b_1R_1c_1S_1d_1T_1d_2^{-1}S_1c_1^{-1}R_1b_3^{-1}$
$Q_1a_2^{-1}P_1 = 1;$

$P_1a_2Q_1b_2R_1c_2S_1d_2T_1d_3^{-1}S_1c_2^{-1}R_1b_1^{-1}$
$Q_1a_3^{-1}P_1 = 1;$

$P_1a_3Q_1b_3R_1c_3S_1d_3T_1d_1^{-1}S_1c_3^{-1}R_1b_2^{-1}$
$Q_1a_1^{-1}P_1 = 1.$

As before, P, Q, R, S, T are all branch points of deficit $3 - 1 = 2$.

$\chi(\tilde{S}) = 3.\chi(S^2) - 5.2$

$= 6 - 10$

$= -4.$

$\tilde{S}$ is closed and orientable, and so must be a three-holed torus.

SOLUTIONS FOR UNIT 7

Section 7.1

1. (i) P -1, Q 0;

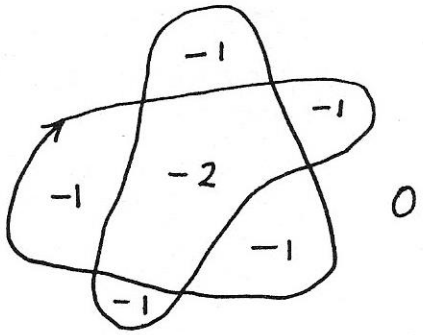
   (ii) P -1, Q 1, R 0;

   (iii) P -2, Q -1, R 0;

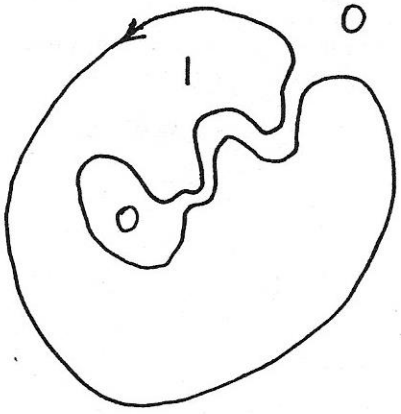
   (iv) P -2, Q -1, R 1, S 0.

2. The answers are as follows.

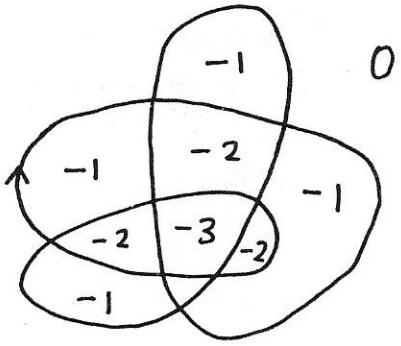
   (i)



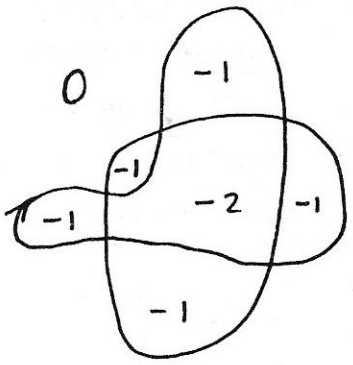
   (ii)



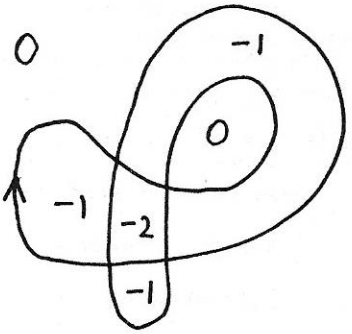
   (iii)



   (iv)

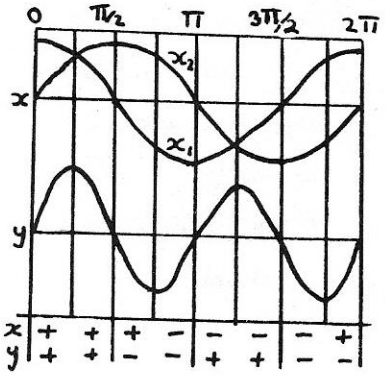


   (v)

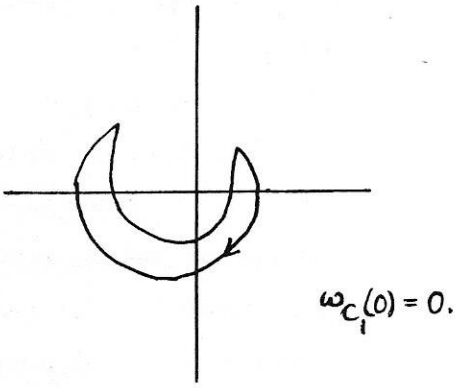


Section 7.2

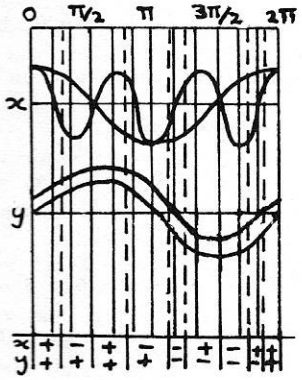
1. (i) We draw the graphs of $x_1 = \cos t$, $x_2 = \sin t$, and $y = \sin 2t$, paying most attention to the points where $t = \frac{k\pi}{4}$, $k = 0, 1, \ldots, 8$.



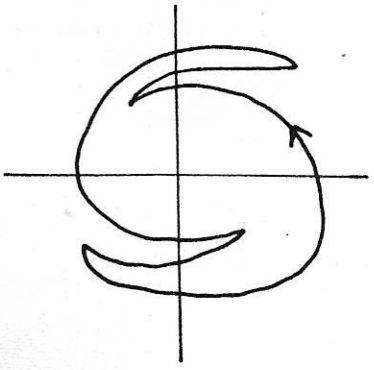
We deduce that $x = x_1 + x_2$ changes sign at $t = \frac{3\pi}{4}$ and $t = \frac{7\pi}{4}$, and y changes sign at $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$. So the curve is qualitatively like this:



(ii) We draw the graphs of $x_1 = \cos t$, $x_2 = \cos 3t$, $y_1 = \sin t$, and $y_2 = \sin t + \frac{1}{4}$, paying particular attention to $t = \frac{k\pi}{6}$, $k = 0, 1, \ldots, 12$.



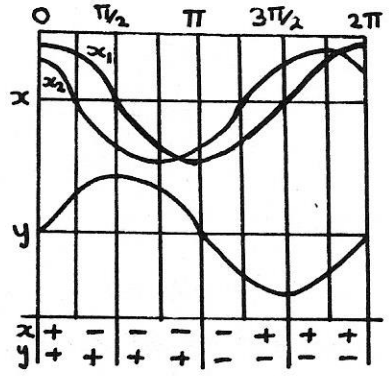
The curve behaves like this,



and $\omega_{C_2}(0) = 1$.

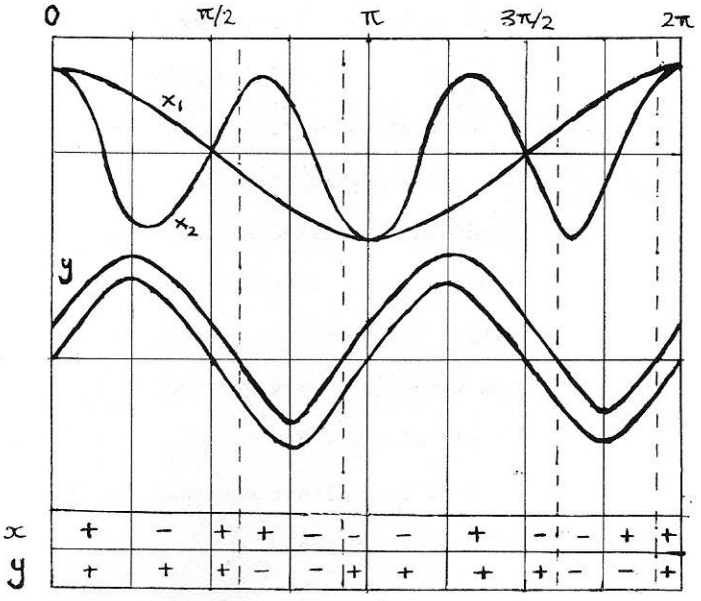
(iii) Since $x > 0$ for all t, $\omega_{C_3}(0) = 0$.

(iv) We draw the graphs of $x_1 = \cos t$, $x_2 = \cos(t + \frac{\pi}{4})$, $y = \sin t$, paying particular attention to $t = \frac{k\pi}{4}$, $k = 0, 1, \ldots, 8$.



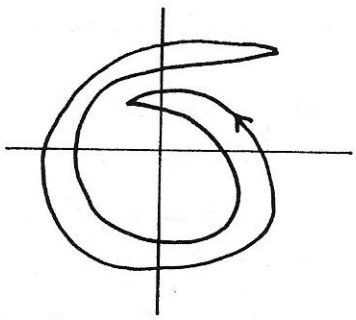
We deduce that $\omega_{C_4}(0) = 1$.

(v) We draw that graphs of $x_1 = \cos t$, $x_2 = \cos 3t$, $y = \sin 2t + \frac{1}{4}$, paying particular attention to $t = \frac{k\pi}{6}$, $k = 0, 1, \ldots, 12$.



We can plot the changes of sign in x and y as indicated, and draw C_5

schematically thus



$\omega_{C_5}(0) = 0.$

Section 7.3

1. On $C_{\frac{1}{2}}$, $|z| = \frac{1}{2}$, set

$f(z) = z^3 + z$ and $g(z) = 1$.

$|f(z)| = |z^3 + z| \leq |z^3| + |z| = \frac{1}{8} + \frac{1}{2} < 1 = |g(z)|$.

So $V_{C_{\frac{1}{2}}}(g) = V_{C_{\frac{1}{2}}}(f+g) = 0$.

So there are no zeros of $z^3 + z + 1$ inside $C_{\frac{1}{2}}$. On C_2, $|z| = 2$, set

$f(z) = z + 1$, $g(z) = z^3$.

$|f(z)| \leq |z| + 1 \leq 3 < |g(z)| = 8$.

So $V_{C_2}(f+g) = V_{C_2}(g) = 3.2\pi$.

So there are 3 zeros of $z^3 + z + 1$ inside the circle $|z| = 2$, and so 3 inside the specified annulus.

When $z = x > 0$, $z^3 + z + 1 > 0$, so there are no positive real roots.

When $x = -\frac{1}{2}$, $x^3 + x + 1 > 0$; when $x = -1$, $x^3 + x + 1 < 0$,

so there is at least one real root between $-\frac{1}{2}$ and -1. Since

$\frac{d}{dx}(x^3 + x + 1) = 3x^2 + 1 > 0$, $z^3 + z + 1$ is strictly increasing along the real axis and so can have only 1 real root; so it has exactly 1.

To locate in which quadrants the complex non-real roots lie, consider the paths

$C_3 = \{z = iy: -2 \leq y \leq 2\}$ and

$C_4 = \{z: |z| = 2, \mathrm{Re}(z) \geq 0\}$.

Along C_3, $F = z^3 + z + 1 = -iy^3 + iy + 1$ and $V_{C_3}(F) = \eta_1$, $|\eta_1| < \pi$.

Along C_4, $V_{C_4}(F) = \frac{3}{2}.2\pi + \eta_2$, $|\eta_2| < \pi$.

So the total variation of F along $C_3 + C_4$ lies between π and 5π. It must be either 2π or 4π, and it cannot be 2π since that would mean the contour enclosed a unique zero of F and we are looking for a complex conjugate pair. So the non-real roots of $z^3 + z + 1$ lie in the first and fourth quadrants.

2. On C_1, $|z| = 1$, set $f(z) = z^3 - z$, $g(z) = 4$.

$|f(z)| \leq |z^3| + |z| = 2 < 4 = |g(z)|$.

$V_{C_1}(g) = V_{C_1}(f+g) = 0$, by Rouché's Theorem, and there are no roots of $z^3 - z + 4$ inside C_1.

On C_2, $|z| = 2$, set $f(z) = -z + 4$ and $g(z) = z^3$.

$|f(z)| \leq |z| + 4 = 6 < 8 = |z^3| = |g(z)|$.

So $V_{C_2}(g) = V_{C_2}(f+g) = 3.2\pi$, and by Rouché's Theorem there are 3 roots of $z^3 - z + 4$ inside C_2. So all the roots of $z^3 - z + 4$ lie inside the annulus $1 < |z| < 2$. The rest of the argument proceeds exactly as in (1) above, and we omit it.

3. On C_1, $|z| = 1$, set $f(z) = z^4 + z$ and $g(z) = 4$.

$|f(z)| \leq |z^4| + |z| = 2 < 4 = |g(z)|$.

So $V_{C_1}(f + g) = V_{C_1}(g) = 0$, and by Rouché's Theorem no roots of $z^4 + z + 4$ lie inside C_1.

On C_2, $|z| = 2$, set $f(z) = z + 4$, $g(z) = z^4$.

$|f(z)| \leq |z| + 4 = 6 < 16 = |g(z)|$.

So $V_{C_2}(f + g) = V_{C_2}(g) = 4.2\pi$, and by Rouché's Theorem all 4 roots of $z^4 + z + 4$ lie inside $|z| = 2$.

Consequently all the roots lie in the annulus $1 \leqslant |z| \leqslant 2$.

We now show $z^4 + z + 4$ has no real roots.

When $z = x \in \mathbb{R}$, $x^4 + x + 4$ has slope $4x^3 + 1$. The only (real) stationary value is at $x = \sqrt[3]{-\frac{1}{4}}$ and is a minimum of $((\sqrt[3]{-\frac{1}{4}})^4 + \sqrt[3]{-\frac{1}{4}} + 4) > 0$. Hence F has no zeros on the real axis.

We now show $z^4 + z + 4$ has no purely imaginary roots $z = iy$. For $y^4 + iy + 4 = 0 \Rightarrow y^4 + 4 = 0$ and $y = 0$, which is impossible.

To show one root lies in each quadrant it is enough to show exactly one root lies in the first quadrant, because then its complex conjugate lies in the 4th quadrant and the remaining 2 roots also form a complex conjugate pair with one in each quadrant.

Consider $C_3 = \{z: z \text{ real}, 0 \leq z \leq 2\}$,

$C_4 = \{z: z \in C_2, \text{Re}(z) \geq 0, \text{Im}(z) \geq 0\}$, and

$C_5 = \{z: z = iy, 0 \leq y \leq 2\}$.

$V_{C_3}(z^4 + z + 4) = 0$.

$V_{C_4}(z^4 + z + 4) = \frac{1}{4}.4.2\pi + \eta_4$, $|\eta_4| < \pi$.

$V_{C_5}(z^4 + z + 4) = \eta_5$, $|\eta_5| < \pi$.

So $V_{C_3 + C_4 + C_5}(z^4 + z + 1)$ lies between 0 and 4π.

It must therefore be 2π, and exactly one root lies in this quadrant, as claimed.

Printed in Great Britain by The Open University